Andy Blunden “Labour Review” February 1984

Dialectics and Mathematics

WITH the publication in 1983, the centenary year of Karl Marx, of the first English edition of Marx’s *Mathematical Manuscripts,* the task of continuing Marx’s work in this field is squarely posed. The manuscripts indicated Marx’s main ideas on the nature of mathematical knowledge in the form of a study of the foundations of calculus. The essence of the manuscripts does not so much lie in what Marx had to say about calculus, but in the method by which he carried out the analysis.

It must be emphasised that Marx’s study of calculus and the manuscripts which have come down to us, cannot be dismissed as peripheral or outdated. When taken in the context of the development of Marx’s ideas as a whole, every aspect of the manuscripts is seen to be significant.

The central position that the understanding of the nature of mathematics has played in the entire history of philosophy, the growing importance of mathematics within natural science as a whole prove that a Marxist analysis of the problems of the philosophy of mathematics cannot be postponed. Throughout the latter part of his life, Marx studied mathematics continuously, and regarded a knowledge of mathematics as essential to ‘a conception of nature which is dialectical and at the same time materialist’. (*Anti-Dühring,* p. 15)

Marx’s study of calculus was motivated, not by the intention of applying it in political economic work, nor for the advancement of mathematics, but in order to sharpen the weapons of his dialectical materialist method in the course of the resolution of philosophical problems that were presenting themselves to mathematicians.

It is when we move to a study of contemporary work on the philosophy of mathematics that the significance of what Marx is doing in the *Manuscripts* becomes clear. It is not so much in relation to problems in the foundation of calculus, which have progressed immeasurably over the past century, but in relation to the most fundamental questions that the *Manuscripts* are most fruitful.

It is precisely because Marx’s ideas on these questions merged completely with his ideas developed in his major philosophical and political works that they are not spelt out by Marx, but are demonstrated in his method of attack, or in apparently inconsequential asides.

It is my intention here to clarify one or two of the main problems in the philosophy of mathematics, indicating the direction in which further work must lead. It is not possible at this stage to indicate the significance of this discussion for either the development of mathematics itself or the development of dialectical materialism as a guide to revolutionary practice, though it is certain that the significance of such fundamental questions must, in the course of time, become clear.

Central to the crisis in the philosophy of mathematics is the question of the nature of the existence of mathematical entities, and how these entities, which have no sensuous or empirical existence in themselves, are able to reflect nature and occupy a central position in natural science which clearly gives us an adequate picture of objective nature. The emergence of contradictions and paradoxes in mathematical logic has also played a central role in the crisis of mathematics.

In this article, the main object will be to demonstrate the importance of dialectics as a theory of knowledge and Logic of science, in opposition to the metaphysical outlook that is universal among mathematicians.

The modern history of the philosophy of mathematics is closely associated with the study of the foundations of mathematics, by which we mean the attempt to formulate mathematical notions and principles from which the whole of the rest of mathematics may be derived mathematically.

This process was initiated during the 1840s following an explosion in the development of diverse fields of mathematics which urgently demanded some sort of unifying theory. At this time George Boole noticed the similarity between formal logic and arithmetic and on this basis developed the algebra of logic, which for the first time allowed mathematics to study itself. From this time forward the philosophy of mathematics has been dominated by one or another form of Kantianism.

Although empiricism has been the predominant tendency among natural scientists generally, empiricism is so clearly a hopeless theory of knowledge for mathematics that empiricism has occupied only a minor position in this arena.

Although fraught by all the contradictions inherent in Kantianism, and passing through a number of important phases, the philosophy of mathematics has remained imprisoned within Kantianism, and has never been able to incorporate the gains of Hegel’s philosophy. Hegel’s derisory attitude to mathematics has been reciprocated by the most important mathematicians.

A study of the philosophy of mathematics must yield for Marxism then a decisive victory in the struggle against Kantianism, the home of most anti-Marxist revisionists.

The most important sources, apart from classical and contemporary mathematical writings, and Marx’s *Mathematical Manuscripts,* are Marx’s *Capital* especially the first three sections of Volume I (1. The Two Factors of a Commodity: Use-value and Value; 2. The Two-fold Character of Labour Embodied in Commodities; and 3. The Form of Value), and some contemporary Soviet works, especially those of E. V. Ilyenkov, which make little or no reference to the mathematics, but which provide important elements for our analysis.

One of the most important driving forces in the crisis in the foundations of mathematics has been the appearance of formal logical contradictions in set theory and formal logic. In common with all bourgeois science, all mathematicians hold that avoidance of contradiction is a fundamental characteristic of mathematics. While this position is clearly inadequate, it would be stupid to simply dismiss it. Suffice it to say that if we were to include the statement ‘2= 1’ in with the existing statements of arithmetic, the whole formerly useful structure would crumble to a useless mass of indeterminacy.

The point is to develop the science of contradiction - dialectics -and in application to mathematics, show how formal contradiction is one special case of contradiction. The point is not to avoid contradiction, but to learn how to handle contradiction, how to utilise contradiction as the ‚principle of all self-movement’, how to **discover** contradiction within the essence of objects in order to consciously build up and concretise new concepts in the course of the resolution of these contradictions.

Does this mean that dialectics is a theory of development, leaving formal logic intact with its avoidance of formal contradiction: ‘contradiction’ in the sense of the word for dialectics being something different? Certainly not.

It is true that dialectics is a theory of development, that is, a theory of knowledge, while formal logic can never be a theory of knowledge , but dialectics is also dialectical Logic, a guide for the deduction of new, more concrete, truths.

The dual character of mathematical entities

ONE of the problems confronting formal logic is to explain how the exchange of one statement for another, equivalent to it, leads to knowledge that was not previously present. This development can only be explained by discovery of the contradiction present within each statement. Rigorous application of the formal-logical ban on contradiction would lead to a denial of the development; dialectical logic not only explains how the new knowledge arises but indicates its path.

Bourgeois political economy faced the same problem in explaining how profit arose in the course of the exchange of commodities of equal value. Ricardo's application of the labour theory of value to resolve this problem led to contradictions. Attempts by his followers to rid political economy of these contradictions destroyed the scientific content of, Ricardo's work; Marx, on the other hand, by conscious application of dialectics created *Capital*.

What made possible Marx's dialectical elucidation of the laws of motion of capitalism, was his discovery of contradiction in the essence of the simplest and most universal relation of capitalist society, the exchange of commodities - the dual character of the commodity itself, both use-value. and exchange-value. Marx showed how the whole of the capitalist mode of production unfolds out of this germ -how the unity of opposite, conflicting tendencies within the commodity manifested itself in the conflict between the capitalist mode of production and the means of production, leading to the historical crisis of the capitalist system.

Mathematical entities - that is, those mathematical concepts which are referred to in the mathematical language in the same grammatical sense as we would refer to things, externally existing objects - are not mental images, in the sense of a sensuous picture derived from experience.

For instance, the concept of 'triangle' (even if psychologically it may be associated with a particular visual image) is not the same as any sensuous image, and can never, as such, be the object of experience. Likewise, '3' can never be experienced. Neither 'triangle' nor '3' can have an independent existence in the material world. There can exist only triangular objects and objects of number three in the material world. The 'triangle' and the '3' can never exist prior to or separately from such objects and in material identification with the countless other properties and interconnections of the objects involved. They do, however, have their basis in nature.

It does not make any difference whether the entity, or concept, is denoted by a word of the English language or by a symbol, such as a natural number or a symbol of set theory. The meaning of the symbol resides in its interconnection with the whole of human practice as a part of nature.

Each such symbol is the repository of a grain of human knowledge, connected in one way or another with the objective practice of man changing nature in accordance with nature's laws. Each entity, symbol or concept is a nodal point in this network, relating man and nature.

The social division of labour, giving rise to the separation of theoretical practice and material-practical activity, obscures from sight the million-fold connection between mathematical concepts and nature, and mystifies the laws governing this relation.

The meaning of a symbol is first how it is used in theoretical practice, the specific social activity of mathematicians, but ultimately, how it is used within the whole labour process. The various symbols, equations etc., which are critically assimilated by the mathematician as he learns how to use them, are the bearers of a whole accumulated history of human labour, and lose all their meaning if separated from that practice.

This social practice, taken as a whole, existed prior to the work of any individual. This may create the illusion of the existence of mathematical entities prior to and independently of mankind. Although these entities have a basis in nature, they do not exist outside of the labour process which gives rise to mathematics as a special science.

The mathematical entity is an external object. The child learns the meaning of '3' in the course of learning how to use this (spoken) word in his intercourse with other people, in learning the art of social life. In the process the child learns how to abstract from the external world sets of objects and to recognise and determine 'how many' they are. At a later stage he learns how to operate with typographical symbols, carrying out specified operations with them, solving set problems with them, and eventually learns, if he is lucky, how material problems may be resolved with the use of such symbols.

Mathematical symbols facilitate the recognition of various relationships, concepts and possible transformations by means of a variety of their formal properties - abbreviation of words to their first letter, affixing of indices, particular spatial juxtaposition, etc - but the manner in which this is done is irrelevant to us: assuming that is, that we recognise that they are tools, man-made, and necessarily possessing properties appropriate to their use.

The idea that mathematics is the art of operating with these symbols is called Formalism. Formalism however, denies that the symbols have any meaning or that statements made up of them can possess truth. Formalism recognises only 'validity' - that is, simply, consistency - within a strictly-defined set of 'regulations for use'.

It is well-known, however, that mathematical symbols act as symbols for something else (as do words of whatever kind) and it is by this fact that their use acquires social significance. Certain symbols, such as the natural numbers 1, 2, 3, ... have an immediate established practical significance. It is taken for granted that the practical operations required for the counting of a set of objects is well-known to all.

It is in this sense that '$x = 3$' can be deemed the answer to the question '$x + 7 = 10$.' The laws for combining such operations as counting and measuring have been empirically established in the course of the long pre-history of mathematics. Since operations with these 'constants' alone cannot constitute mathematics, let us leave aside a deeper analysis of such concepts as number (cardinal and ordinal numbers) and look at the point where mathematics itself begins.

What is always something of a mystery for the newcomer to mathematics is how it is possible to carry out calculations with letters instead of numbers. Indeed, the historically first method for the solution of mathematical problems, in both the Near and Far East, the method of False Position, did not use such symbols for an unknown quantity. Only definite numbers were used; the true solution being found by adjustment of a first arbitrary trial.

The discovery that symbols may be manipulated as if they were definite numbers is a decisive turning point in the history of mathematics and the use of such symbols is universal and characteristic of mathematics. It is the source of the problem-solving utility of mathematics and is central to the ability of mathematics to generate new concepts but also the source of all the paradoxes, philosophical problems and contradictions of mathematics.

The point that is frequently overlooked in relation to the mathematical entity is that it has essentially a dual character, it is always essentially ambiguous - representing at the same time two opposite things.

Referring to Taylor's theorem, Marx says in his *Mathematical Manuscripts:* 'In place of the real successive derived functions of x only the derivatives are represented, in the form of their symbolic equivalents, which indicate just so many strategies of operations to be performed, independently of the form of the function ...' (p. 110)

Or, on p.21-2: 'Now, what are the corresponding "derivatives" of the symbolic differential coefficients du/dx, dz/dx ? ... answered if one substitutes arbitrary original functions of x for u or z. For example $u= x^4$; $z= x^3 + ax^2$. Thereby, however, the symbolic differential coefficients du/dx, dz/dx are suddenly transformed into **operational symbols**, into symbols of the process which must be carried out with x^4

and $x^3 + x^2$ in order to find their “derivative,” and thus already finished, the symbolic differential coefficient now plays the role of the symbol of the operation of differentiation which is yet to be completed.’

Even the simplest mathematical entity, such as x represents **both** the series of already well-known operations by which x is to be derived from any given starting point **and** a definite number, which can be added, multiplied etc. All mathematical entities have this dual character, which may be built up in successive ‘layers’. For instance, f(x) represents both the operations to be performed on any given x, and a definite value, which may be the subject of the usual operations of algebra. That is, they have both a qualitative and a quantitative aspect.

The utility of the thing in facilitating the perception of the operations which are to be performed in any concrete case turn it into an operational symbol. But quantitative determination is always pre-supposed since only a definite object may be measured, only a definite number subject to the specified ‘mapping’ etc. Thus, the qualitative value of the entity can only be realised in execution of the operations indicated, leading to a definite quantitative result. These symbols are the substantial bearers of the mathematical knowledge of past generations.

When we bring two such objects into relation, such as in $y = x$, we say that the same value is represented in two different things; each is therefore equal to a third which is neither the one nor the other, although each is, independently of the other, reducible to this third. This third, their value, constitutes their unity. This unity does not arise out of nature, but out of society, out of the socially determined practices of the measurement of number, or external form of another kind.

Although the different mathematical entities clearly represent successively more complex modes of quantitative cognition, marking the various qualitative levels incorporated in the build-up of quantitative science, it is clear that each is reducible to the several simplest modes of formal cognition - of order, structure, number etc.

It is true that, especially in higher mathematics, the symbol for an operation is often separated from its value, which remains only implicit. In some cases, the facility of numerical manipulation is not thereby lost, or may be restored in a new form, giving rise to a qualitative new system of relations.

On the other hand, a collection of numerical data, to be sure represents new quantitative knowledge, but does not yet constitute mathematics. In order to enter mathematics the symbol for a quantity must also be a symbol for something else, already known, which gives to the concept its social significance, its internal conflict, and thus its movement into something else. But similarly, unless it is the symbol for some process of determination, it cannot have a value.

The genesis of mathematical concepts

IN HIS *Mathematical Manuscripts,* Marx deliberately confined himself to consideration of the simplest possible functions. Beginning with the differentiation of $y = ax$, for which none of the difficulties of vanishing differences etc have any need to arise, he is able to rigorously outline the main concepts, and demolish the mystical notions of infinitely small quantities etc current in his time. This method is no accident, but is line with the method of *Capital.*

In his analysis of capitalism Marx showed that the relationship of the exchange of commodities was the simplest, most universal and essential relation of capitalism; all other social relations are conditioned by this relation, determined by it. And this is despite the fact that simple exchange of commodities almost never occurs as such in modern capitalist society, but is only to be found in primitive societies where commodity production is peripheral and undeveloped.

By means of the most thorough analysis of the commodity relation based on a painstaking dialectical analysis of all the empirical data, Marx is able to establish all the most fundamental concepts - value, use-value, exchange, labour-power, etc - and disclose the necessity of the processes unfolding from this germ.

Likewise, in the *Mathematical Manuscripts,* although the simple algebraic functions with which Marx deals form only a minor part of modern analysis they are the simplest and most basic. Marx's interest in MacLaurin's theorem clearly relates to the need to elucidate the relation of the more general functions to these algebraic series.

Marx's method of differentiation is not generally applicable outside this elementary area, but he is able to rigorously derive the more advanced concepts necessary for further development. If we were to set out to establish a foundation from the beginning applicable to the most general functions, we would have to introduce principles and concepts the origin of which was completely mysterious.

The process is similar to the generation of the real number field from the integers by the 'generic' method, in which the concept of number is successively expanded by incorporating the contradictions arising from the inversion of operations proper to the primitive number field alone.

This is an instance of dialectical logic, and incidentally, contains the essence of the 'axiomatic method' of modern pure mathematics. For instance, a knowledge of addition allows the solution of such problems as $x+7=10$. ie $x=3$; but also allows problems such as $x + 10 = - 7$ to be stated. The solution of this one by means of the extension of the number concept to negative values, allows the solution, $x= -3$. This conclusion cannot be derived, or even justified by formal logic, which can only, *a posteriori,* prove the consistency of the conclusion.

Marx's method, as in *Capital,* was a unity of the logical and historical methods of investigation. The generic method should not be confused with abstract historicism which starts from an arbitrarily chosen starting point, and follows only the empirically most general features of the object in its historical sequence. The object lacks internal contradiction, and its movement is therefore mystified.

By contrast, dialectical materialism -concrete historicism - recognises that the premises of a concept develop in the opposite way to the concept itself. The concept is at first abstract and ill-defined. Each new historical step forward reveals what lay beneath, brings to the surface what was previously present only potentially.

The logical and historical processes are thus opposites. The determination of the true, objective point of origin of a concept requires a concrete, dialectical definition of it. Only in this way can a study of historical processes provide objective knowledge of the movement of the concept.

Marx showed how Newton and Leibnitz **began** with the differential, which was introduced without explanation. He traces the successive stages in the development of calculus, which each sought the explanation of the differential, and the resolution of problems arising from the earlier stage; as methods of calculation however, each was less powerful than the earlier. Three hundred years had to pass before the historical process negated itself and the original 'mystical' method of differentials could be logically substantiated by non-standard analysis and category theory.

These points have been made in order to emphasise that in order to understand the nature of mathematical entities we must start with the simplest, that is, the simple unknown quantity, x. Here the dual nature of the entity may be simply and rigorously proved - it is both a number and not a number. It is not the place here to trace the elaboration of more complex mathematical entities, but simply to indicate a beginning. It is not important whether x is the symbol for a number, a set, an operator, logical proposition, point, space, group, category ...

Dialectical logic is able to proceed from here, aided by an historical study to establish the interconnections and origins of new concepts such as from algebraic expression to function to operator etc. Formal logic cannot generate new concepts any more than a knowledge of the laws of mechanics are sufficient to design a motor car. While dialectical logic sees contradiction as the source of new, more general concepts, formal logic denies the validity of the extension of a concept absolutely, deplores contradiction, which it hopes to avoid by endless amendment of the initial abstract premises.

Essential to the elaboration of a dialectical concept of mathematics is the writing of a dialectical history of mathematics, taking Marx's logical-historical analysis of calculus as a model. The meaning of a concept can only be grasped by identifying the problems which were being grappled with at each historical juncture, and which gave the objective impulse necessary for the origination of a new concept.

'The Inversion of the Method'

THE published *Mathematical Manuscripts* of Marx show how Marx worked and re-worked his study of what he called the 'inversion of the method' in the transition from simple algebra to calculus. This concept is the central dialectical idea in the manuscripts, and is essential to understanding the dual character of mathematical entities, and the dialectical structure and development of mathematics as a whole. Whereas formal logic can grasp the difference between concepts, the essential thing for dialectics is to grasp the. transition from one to the other.

In the first manuscript, 'On the Concept of the Derived Function', Marx sets out his method of differentiation, deriving dy/dx, first in the case of y= ax, where the ratio of finite difference is identical with dy/dx Successively more complex cases are considered, including the exponential function, by means of its power series expansion, in which Marx derives dy without recourse to the method of approach to a limit, having relied on definition of continuity at $\Delta x = 0$.

He proves that dy/dx is the symbol for a function, just as is f(x), but which also indicates which operations are to be applied to any defined function of x. He notes that all functions could be treated in this way, from first principles, but this he describes as a 'damned useless mass of details.'

Nowhere up to here does Marx have recourse to the concept of a differential having an existence independent of the differential ratio.

In the second manuscript, 'On the Differential', Marx uses the product y = uz to derive the product rule for differentiation. The point of interest here, as Marx emphasises in a supplement, is that the derivatives du/dx and dz/dx appearing now on the right-hand side of the equation as 'symbolic differential coefficients without corresponding equivalents' thus become 'independent starting points and ready-made operational formulae.' Here, Marx says, 'the algebraic method reverses itself into the differential method.'

The symbols of calculus arose out of algebra as symbols for operations which have already been carried out algebraically. What appears then is the calculus, as an independent method of calculating, operating on its own ground. The operations of calculus are now assumed to be well-known. The symbols represent operations yet to be carried out in any concrete case. Operating on its own ground - with its own characteristic symbols and rules for their combination, which are taken for granted and do not have to be derived algebraically.

In deriving the quotient rule from the product rule Marx illustrates in the supplement how calculus generates from itself new operational formulae. Thus, as soon as the differential formula appears the movement has inverted itself, in order to derive new algebraic functions. From algebra to calculus, and back to algebra, but with calculus now acting as an independent starting point.

Since, from the standpoint of algebra, the differentials, dy and dx, can have no independent existence, their appearance in operational formulae of calculus marks the moment of this inversion - being symbols for operations, assumed well-known, but yet to be carried out, in any specific case.

Marx is here developing in a new area a dialectical idea with which he had been concerned from an early age -indeed the first work in Volume III of the Marx-Engels Collected Works (the first two contain their 'early work'), a critique of Hegel's *Philosophy of Law*, concerns this concept. The whole theory of fetishism, alienation, the social origins of consciousness, dialectical transition, are contained here.

The 'inversion' of the conflict within the forces of production of capitalism, the reversal of roles in the class struggle was of course the transition which was the driving force for Marx's life. But Marx gives us here an indication of how a dialectical conception of mathematics is to be built up.

In the traditional presentation of Newton and Leibnitz calculus appears to proceed from itself - an independent force standing above algebra. This is the mystical conception, just as is the illusion that law determined social practice, rather than being an expression and product of prior social practice; social practice determining itself, by means of law.

While calculus does become an independent realm, operating on its own ground, its symbols are only meaningful because they indicate, symbolise, operations to be carried out algebraically, and dy/dx, the fundamental symbol of calculus, is the symbol both for operations to be carried out, but also the symbol for a function namely the derived function of $y = f(x)$.

Each qualitative development in mathematics is marked by the appearance of qualitative new symbols, denoting new concepts having relative independence on their own ground, but either having as content the transition or movement between entities of a lower order, or revealing the content of other concepts by denoting transitions hitherto not grasped.

The transition from the field of functions to the field of differential operators is a transition from quantity to quality. While all thought is abstract necessarily, since it can grasp nature only incompletely, truth on the other hand must be concrete. Mathematics is the science of quantity. The category of quantity is at first simple magnitude; but with the development of mathematics qualitative leaps take place in which the transition between quantities is grasped as a qualitative moment. The category of quantity thus cannot stand aside from its ' polar opposite quality but incorporates it within itself in its movement through contradictory concepts unifying opposite qualitative aspects, approaching measure by its own characteristic path.

Mathematical abstraction . differs from other scientific thought in that it does not abstract the essence of an object from the whole of its interconnections but abstracts an external relation from the object itself. Like all scientific thought it approaches the concrete by unifying diverse abstractions, but unlike the concrete sciences it does not aim to begin with the essential quality of objects, but sets out from the opposite pole - inessential quality.

The process of demonstration of similarities in structure between various branches of mathematics at any given epoch, filling out the abstract concepts with a more concrete content, the elaboration of the ramifications and interconnections between them, prepares the way for a new qualitative leap.

The inversion of the method manifests itself in the objective historical process in an inverted form. The transition only appears after it has already taken place in a mystified form. The fullest development of calculus was necessary before calculus provided the material necessary to clarify its origin from algebra. The most fundamental concept emerges only at the end of the day. The further development of science then can only be carried out consciously to the extent that the historical meaning of its concepts can be grasped, and this is as true of mathematics as it is true of political economy, history, etc.

The idea that the essence of a new mathematical concept lies in the inversion of the method of its derivation proves that mathematical labour must move in two directions. The one-sided, formal-logical conception of deductive proof, which leaves out the reverse motion of application of the new concept to resolve the problems out of which it arose, cannot grasp how mathematics contains new knowledge **at all**.

The Equivalence relation

ONE thing that particularly strikes the eye when reading Marx's *Mathematical Manuscripts* and which seems particularly curious to the mathematicians is the interest Marx shows in the differing roles of the left and right sides of the equals sign. He refers to the 'symbolic' side and the 'algebraic' side and to the 'initiative' shifting from one side to the other, etc. While to the formal-logician, equals is 'symmetric' and therefore an equation may be written one way or the other indifferently, Marx had previously studied the equivalence relation in its objective and dialectical movement in relation to the exchange of commodities.

In studying the way mathematicians in fact use the equals sign, he is tracing the real movement of cognition. In doing so he is able to draw out relationships, and detect changes that escape the notice of the metaphysician.

Mathematics is a social practice. Mathematicians **use** the tools of their trade, such as the equation-form, that they inherit from the past. Marx's conception of mathematics as human practice is far richer than the conception of formal logic, for whom all possible truths existed from the beginning, and for whom there is no transition.

Marx's Capital contains an exhaustive exploration of equivalence, and Marx drew on Hegel's *Science of Logic* in abstracting his political economic theory from a study of contemporary capitalism and its historical predecessors. There is no point in trying to paraphrase Marx's most outstanding work. Anyone wishing to learn about dialectics in mathematics as elsewhere must read at least the first three sections of Chapter 1 of *Capital*. The identity of identity and difference, that like cannot be exchanged for like, is proved concretely: how the determinations of reflection identity, variety and opposition pass over into contradiction ...

Let us look more closely for a moment at the differing concepts of equivalence as between formal logic on the one hand, and dialectics on the other.

Mathematics defines an equivalence relation as one having reflexivity (A = A), symmetry (if A = B then B = A) and transitivity (if A = B & B = C then A = C). Let us look at each formal-logical component of equivalence in turn.

(It must be said however, that '=' is the kernel of dialectics in mathematics, and a full study of its nature would require a comprehensive study of dialectics. Clearly, a preliminary article such as this cannot even touch on that task.)

Reflexivity: The Law of Identity, **A= A**. Outside the context of the definition of the symbol, =, this law is a useless tautology which leads nowhere - except in so far as 'A on the left' is not the same, but the opposite of 'A on the right.'

That is, the meaning of the law is the identity of opposites - the statement that every single concept contains two opposite sides. In the formal-logical interpretation that everything is equal to itself, it is either a barren and useless statement leading nowhere, if taken abstractly, or, if applied concretely, wrong and one-sided, since A also not = A.

Symmetry: The law of Difference, B = A means A = B. While this law obviously has its place in the mechanics of syllogistic reasoning, handling equations, it would be entirely wrong to characterise this law as stating that the left and right-handed versions of an equation are identical or interchangeable.

Marx pays great attention to this differing of the roles of the left and right sides of equations. Equations are **used** by mathematicians as instruments for cognition, and their meaning resides in their use in true social practice, not in their abstract definition.

If we translate '=' as 'is', we would correctly conclude that 'x = 7' does not imply '7 = x', although within a strictly -defined context such implication would have formal truth. The falsity lies in misrepresenting the movement, from x to 7. Formal interpretation of this law, of course, denies the movement of cognition, since to formalism all the propositions of a theory are contained within its premises.

Transitivity: This law is the most powerful instrument of equivalence since it allows two concepts, A and C, to be transformed into one another that were formerly connected only by the intermediary, B. It is precisely this that formal logic ignores since it denies that A = C has a content differing from A = B & B = C.

But this movement has included the transition of B into its opposite, from predicate to subject, and the deduction is therefore true only to the extent that B is self -identical. The law of transitivity therefore can have absolute truth only abstractly, and it can be seen that if it is to reflect reality each transition must have an objective basis, so that the movement from A to C mirrors a real transition. Thus the formal conception which denies this movement misses the objective significance of the equations.

In summary then, it is not so much that the law of equivalence is wrong, but that the formal-logical conception of it is one-sided. It fails to understand how new knowledge can be derived from the production and exchange of .logically equivalent statements, and prevents an understanding of how mathematical reasoning may form a correct picture of objective reality. A dialectical understanding of mathematics recognises that every equation is a contradiction and the point is to draw upon the theory of knowledge of Marxism in order to discover and grasp contradiction within every object - in order to change it.

Formal logic hoped to find its last refuge in the science which did not pretend to reflect the content - the whole of the interconnections of its objects of study. But we can see that even here not a single step forward can be taken without the statement of a contradiction, without equating two opposite things.

This is because cognition can only move forward if the concepts it uses have a dual character, are contradictory. Every moment of identity thus immediately passes over to a moment of difference, and thus to opposition, the unity of identity and difference. The discovery of properties within an object (for instance those of arithmetic) which do not belong to it (such as with differential operators which may be manipulated as if they were numbers, but they are not numbers) is the essence of the creation of a mathematics of the object.

Thus, it is the dual nature of mathematical objects, excluded by formal logic, which is the source of the development of mathematical knowledge. It is, however, also the source of formal contradictions which have been at the root of the crisis in the foundations of mathematics for a century, and the opposition between structuralism and constructivism.

These problems created a crisis because in order to avoid contradictions, the initial premises and rules of inference were so curtailed that it became impossible to justify what mathematics was actually doing, which in turn, appeared to the formal logicians as a gigantic fallacy. The further progress of formal logical analysis of mathematics has only the more completely eroded the possibility of a formal-logical basis for mathematics.

The appearance of formal contradictions certainly has shown the limitations of formal, abstract and metaphysical thought which refuses to recognise the objectivity of these contradictions. But dialectics seeks by the discovery of contradiction to deepen concepts, so that they more truly reflect the objective laws of the movement of matter.

The contradictory nature of mathematical entities allows valid statements to be made about mathematical entities which have not, or cannot, be constructed - that is, entities which lack the quantitative pole. The constructivist logic of Brouwer and others, resulting from this discovery, rejects the law of the excluded middle as an absolute law. It undoubtedly has merit for this. If rigidly applied though, it prevents the transition to new concepts which can arise out of the contradictions constructivism seeks to avoid.

Gödel 's theorem proving the existence of formally undecidable statements within any consistent theory may provide an important pointer to how the essence of a theory must lie outside of its own bounds. It is, in general, however, interpreted negatively by formalists.

The crisis in mathematics

WE could define the crisis in mathematics as the inability to explain the nature and source of mathematical truth, as opposed to formal validity. The struggle to find the answer to this within mathematics itself is the crisis of the foundations of mathematics.

Mathematical activity is a part of the whole, social and historical process of human cognition, that is to say, the labour process as a whole. Any attempt to cut mathematics off from its interconnection with this whole will necessarily lead to crisis in understanding the connection of mathematics with external nature.

Marx was opposed to any pragmatist or empiricist conception of mathematics. The connection of a mathematical concept with the external world may be extremely indirect, and Marx always insisted on the highest standards of logical precision in order to maintain that connection. Indeed, he subjected reasoning to the strictures of dialectics. Although dialectical concepts are more flexible, their infinitely greater content means that correct dialectical reasoning is infinitely more demanding than formal reasoning. The suggestion by pragmatists and empiricists that materialism favours a looseness in mathematical logic is entirely wrong.

While the essence of mathematics does lie outside mathematics, mathematics must justify its concepts by its own methods.

In the supplement to the second manuscript, Marx translates the product rule for differentiation, which he has just derived, into words. This not only de-mystifies mathematical symbolism, but proves that while the use of formalism is characteristic of and essential to mathematics, the use of symbols does not affect the fundamental relation of mathematical concepts to nature.

All mathematical statements are translatable in this way, although in words they are usually clumsy and less comprehensible, with excessive use of pronouns and technical terms - less ‚graphic'. And it would be a mistake to make a fetish of mathematical formalism. Even the simplest axioms of mathematical theory cannot be stated without recourse to the common language. Where logical symbols are used instead of words they merely act as a code, and lose all their meaning and usefulness if interpreted only formally.

Analysis of mathematical truth then can only be carried out as a part of the analysis of human life-activity as a whole. Philosophers of mathematics have mostly attempted to explain mathematics from itself, though they have not been averse, in the style of reactionary scribblers, to applying their conclusions back to society. We shall briefly review the main trends in the philosophy of mathematics that have been most important over the past century.

Logicism: Leibnitz, Frege, Russell based themselves on the proposition that the whole of mathematics can be deduced from a small number of axioms, which are self-evident, by means of formal logic, whose laws are, in turn, self-evident. The axioms would relate to a number of primitive notions, the basic objects of mathematics, in terms of which all others could be defined. This program has not, and all accept that it cannot, be completed. But mathematics under this view is a part of Logic as a whole, being characterised only by its concern with the specifically mathematical primitive notions. It would be a special science, but share its logical basis with science as a whole.

Logicism did not concern itself with the relation of its primitive notions to the external world, as this was unknowable from the standpoint of logic, and relied on the innate knowledge an individual is deemed to have of Logic, and is thus essentially a part of Kantianism.

Logicism ran into crisis on two fronts. Firstly the foundations of formal logic in science as a whole proved a liability rather than a strength, as the dialectical development of the concrete sciences exposed the inadequacy of formal logic, which now had to be defended as a special device of mathematics. Secondly, since the original self-evident notions led to formal contradictions, their modification negated their self-evidence, and one was left with having to select the premises of mathematics in the light of experience - the negation of Logicism.

Cantor's philosophy was essentially similar, choosing set-theory as a foundation for formal logic, rather than the reverse; both Logicism and Cantorism wound up in the same crisis.

This crisis led to a retreat into the sceptical side of Kant's philosophy with the **Formalism** of Hilbert, who in trying to save mathematics by proving the consistency of its theories, was led to denying the truth or meaning of mathematics altogether. Mathematics concerned only the validity of rules for the manipulation of symbols (not really 'symbols' since they did not symbolise anything else). The connection with or applicability to material reality of these symbols was denied, or deemed inexplicable.

Formalism negated itself through the work of Hilbert's follower, Gödel, who derived meaningful theorems demarcating the limits of formal proof, by making mathematics itself the object of mathematical research.

Gödel himself believed in the objective existence of mathematical entities, and he claimed that it was only this view that enabled him to derive the theorems that his great teacher, Hilbert, had missed.

In the meantime, the **Intuitionist** Brouwer had already begun his attack on Hilbert's Formalism in order to re-establish a basis for mathematics as a science. Brouwer based himself on **intuition**. Mathematical ideas were directly intuited, and only afterwards, imperfectly, communicated to others by means of the mathematical language. This is the position of Kantian bourgeois individualism in its purest form. All discussion of mathematical truth is reduced to a psychological debate on what seems to one individual to be intuitively true.

By trying to negate the scepticism of the Formalists, while at the same time incorporating the lessons of the crisis of Logicism which gave rise to Formalism, Brouwer's intuitionism made an important contribution. Brouwer insisted that mathematical concepts had to be constructed. He opposed the unbridled application of formal logic to objects whose existence had yet to be proved.

One is immediately reminded of Marx's criticism of Newton and Leibnitz's introduction of differentials at the beginning of calculus without explanation. As a result, Brouwer rejected the Law of the Excluded Middle as an absolute law of logic, since a thing had to be, before it could be A or not A.

Platonism - the belief that mathematical entities existed, in Being, prior to their discovery (or 'recollection' in the classical school) by mathematicians - is an ancient philosophy which is compatible with Logicism or Cantorism, but is specifically negated by Intuitionism with its constructivist logic. The move towards Platonism in more recent times is an attempt to escape from the contradictions of Kantianism in the direction of objective idealism. But without dialectics, modern Platonism has moved in the direction of adaptation to the crudest mechanical materialism.

The Intuitionist critique of mathematical logic is extremely negative, rejecting much apparently valid mathematics in the process of avoiding problems arising from the structuralist method. Many mathematicians today say that it is necessary to act **as if** Platonism were true, although of course it isn't. Kant himself moved in this direction.

The problem here, of course, is that mathematical concepts do exist prior to the individual's cognition of them, as products of the social labour of past generations, inherited by an individual when he learns to use the existing repertoire of mathematical symbols and equations, etc. But they do not exist prior to man, independently of social practice, because they are **concepts** - nodal points in the process of human cognition of nature. These concepts are thought-objects, but they do have a basis in nature.

Without triangular objects or objects related in a triangular way, the concept of triangle could not exist. In nature triangles do not exist - only triangular objects. That is, in nature everything is concrete - is connected by infinitely many threads to every other object. Mathematics is the science which studies forms; the relation between objects, abstracted from the objects themselves. That is why fields of mathematics find their application in diverse areas of concrete science, and why long chains of formal reasoning may lead to valid conclusions, which may, conditionally, be applied again to the external world from which they were borrowed.

A concept is not a mental image, and is not derived from experience. Mathematics then has its basis not in immediate sensual experience, or even 'processed' experience, but in the long, historical build-up of concepts based on social practice which successfully changes the world by producing objects which embody our concepts of things, the labour process.

These remarks are intended to indicate the main lines upon which the resolution of the outstanding problems in the philosophy and foundations of mathematics must begin. I have sought, however, to bring to light, and sharpen our understanding of certain aspects of dialectical materialism in the course of this criticism of the bourgeois philosophy of mathematics.